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Linear Algebra and its Applications 346 (2002) 273–291

LINEAR ALGEBRA AND ITS APPLICATIONS

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On block completion problems for Arov-normalized $j_{qq} - J_q$-elementary factors

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Received 8 March 2001; accepted 3 October 2001

Submitted by M. Fiedler

Abstract

The interest in analyzing Arov-normalized $j_{qq} - J_q$-elementary factors is mainly caused by the fact that given a nondegenerate matricial Carathéodory problem there is a unique nondegenerate Arov-normalized $j_{qq} - J_q$-elementary factor which (via linear fractional transformation of the matricial Schur class) generates a description of the solution set. The main goal of this paper is to study canonical block completion problems for Arov-normalized $j_{qq} - J_q$-elementary factors. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 30E05; 47A57

Keywords: Matricial Carathéodory problem; Resolvent matrices; $j_{qq} - J_q$-elementary factors

0. Introduction

The treatment of the matrix version of the classical Carathéodory problem had led to various interesting new subclasses of rational matrix-valued functions. Under some nondegeneracy condition the solution set of a matricial Carathéodory problem can be parametrized by a linear fractional transformation generated by some rational matrix function of a special type. Introducing appropriate concepts of normalization one can construct one-to-one correspondences between nondegenerate matricial Carathéodory problems and particular classes of rational matrix-valued functions. A

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PII: S0024-3795(01)00523-7

first bijection of this kind was constructed by Kovalishina [16] (see also [14]) who used a method of V.P. Potapov. Another type of resolvent matrix for the nondegenerate matricial Carathéodory problem was presented without proof in the work of Arov and Krein [5]. Later it was refound in an alternate way in [11, Part V]. The problem of characterizing this resolvent matrix has led to the notion of so-called Arov-normalized $j_{qq} - J_q$-elementary factors (see [13]). Hereby we mean a particular subclass of $2q \times 2q$ rational matrix functions which turn out to be intimately connected to the signature matrices $j_{qq} := \operatorname{diag}(I_q, -I_q)$ and

$$J_q := \begin{pmatrix} 0_{q\times q} & -I_q \\ -I_q & 0_{q\times q} \end{pmatrix},$$

where I_q stands for the $q \times q$ unit matrix. These matrix-valued functions are defined in terms of the canonical block decomposition into four $q \times q$ blocks. From this it will be clear that the four canonical $q \times q$ blocks are the essential building stones of Arov-normalized $j_{qq} - J_q$-elementary factors. For this reason, it is useful to study the synthesis of these functions on the basis of a prescribed pattern of blocks. A first step in this direction was given in [13] where the lower $q \times 2q$ block row was prescribed. The main goal of this paper is to present a method which enables us to treat several patterns of block completion problems for Arov-normalized $j_{qq} - J_q$-elementary factors. Some of these completion problems could also be fitted into the general scheme of studying block completion problems for $j_{qq} - J_q$-inner functions which was developed in [3,4]. However, the approach below is closer adapted to the specifics of polynomial $j_{qq} - J_q$-inner functions.

1. Some preliminaries

Throughout this paper, let n be a nonnegative integer, and let p, q and m be positive integers. Let $\mathbb{C}$ be the set of complex numbers, and let $\mathbb{C}^{p\times q}$ be the set of all $p \times q$ complex matrices. If $A \in \mathbb{C}^{q\times q}$ and $B \in \mathbb{C}^{q\times q}$, then $A \geqslant B$ (respectively, $A > B$) means that $A - B$ is nonnegative Hermitian (respectively, positive Hermitian). If A belongs to $\mathbb{C}^{q\times q}$, then $\operatorname{Re} A$ and $\operatorname{Im} A$ designate the real part of A and the imaginary part of A, respectively. Further, let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. If A is a $p \times q$ matrix polynomial, then $\check{A}$ designates the matrix polynomial given by $\check{A}(z) := (A(\overline{z}))^*$ for each $z \in \mathbb{C}$, and if $\mathscr{M}$ is a nonempty subset of $\mathbb{C}$, then $\operatorname{Rstr.}_{\mathscr{M}} A$ stands for the restriction of A onto $\mathscr{M}$. If A or B are $2q \times 2q$ matrix-valued functions, then we will work with the $q \times q$ block partitions

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{1.1}$$

respectively. Observe that the function $\check{A}$ has the $q \times q$ block partition

$$\check{A} = \begin{pmatrix} \check{A}_{11} & \check{A}_{21} \\ \check{A}_{12} & \check{A}_{22} \end{pmatrix}. \tag{1.2}$$

If a $p \times q$ matrix polynomial P of formal degree not greater than n is given by

$$P(z) = \sum_{k=0}^{n} a_k z^k, \quad z \in \mathbb{C}, \tag{1.3}$$

then the reverse matrix polynomial $\tilde{P}^{[n]}$ of P with respect to the formal degree n is defined by

$$\tilde{P}^{[n]}(z) := \sum_{j=0}^{n} a_{n-j}^* z^j, \quad z \in \mathbb{C}.$$

An $m \times m$ complex matrix J is called an $m \times m$ signature matrix if $J^* = J$ and $J^2 = I$. Observe that if α is an eigenvalue of an arbitrary $m \times m$ signature matrix, then $\alpha = 1$ or $\alpha = -1$. If p is the multiplicity of the eigenvalue 1 of an $m \times m$ signature matrix J, then we will say that J has the signature $(p, m-p)$. In particular, we will consider the $2q \times 2q$ signature matrices j_{qq} and J_q given above, which are intimately connected to the so-called matricial Schur problem and the matricial Carathéodory problem, respectively (see, e.g., [7,8]). The signature matrices j_{qq} and J_q have the same signature (q, q).

2. On Arov-normalized $j_{qq} - J_q$-elementary matrix polynomials

Let $J^{(1)}$ and $J^{(2)}$ be $m \times m$ signature matrices with the same signature. In this section, we will consider a class of matrix-valued functions whose restrictions onto $\mathbb{D}$ are $J^{(2)} - J^{(1)}$-inner functions, the set of the so-called $J^{(2)} - J^{(1)}$-elementary polynomials. An $m \times m$ matrix polynomial A is said to be a $J^{(2)} - J^{(1)}$-elementary polynomial if the following three conditions are satisfied:

(i) A is not a constant matrix-valued function.
(ii) For each $z \in \mathbb{D}$, the matrix $A(z)$ is $J^{(2)} - J^{(1)}$-contractive.
(iii) For each $z \in \mathbb{T}$, the matrix $A(z)$ is $J^{(2)} - J^{(1)}$-unitary.

Observe that, if k is a positive integer, then the set $\mathscr{E}_k^{(J^{(2)},J^{(1)})}$ of all $J^{(2)} - J^{(1)}$-elementary polynomials of formal degree k coincides with the set of all $J^{(2)} - J^{(1)}$-elementary factors with pole of order k at $z = +\infty$ (see [13, Definition 4]). If $J^{(2)}$ and $J^{(1)}$ are $m \times m$ signature matrices which have the same signature $(p, m-p)$ with $0 < p < m$, then the rank of the leading coefficient matrix of any $J^{(2)} - J^{(1)}$-elementary polynomial is not greater than p (see [9, Proposition 1]), and we will denote the set of all $J^{(2)} - J^{(1)}$-elementary polynomials for which the rank of the leading coefficient matrix is equal to p by $\mathscr{F}_{n+1}^{(J^{(2)},J^{(1)})}$.

Remark 2.1. Let $J^{(1)}$ and $J^{(2)}$ be $m \times m$ signature matrices with the same signature $(p, m-p)$, where $0 < p < m$. Then $A \in \mathscr{E}_{n+1}^{(J^{(2)},J^{(1)})}$ if and only if $\check{A} \in \mathscr{E}_{n+1}^{(J^{(1)},J^{(2)})}$. Moreover, $A \in \mathscr{F}_{n+1}^{(J^{(2)},J^{(1)})}$ if and only if $\check{A} \in \mathscr{F}_{n+1}^{(J^{(1)},J^{(2)})}$.

In his investigations on inverse questions connected with the matricial Nehari interpolation problem Arov [1,2] introduced a concept of normalization of γ-generating (p,q)-type functions. These functions proved to be intimately connected with $j_{pq}-j_{pq}$-inner functions. Using these relations a concept of normalization for $j_{pq}-j_{pq}$-inner functions was proposed in [10]. Among the class of $j_{pq}-j_{pq}$-inner functions full rank $j_{pq}-j_{pq}$-elementary polynomials play a distinguished role. They occur as resolvent matrices of nondegenerate matricial Schur problems. In the case of full rank $j_{pq}-j_{pq}$-inner polynomials can be characterized in terms of the canonical block structure. This observation was the starting point for the definition of Arov-normalization for $j_{qq}-J_q$-elementary factors in [13]. This concept will be essential for our further considerations. For this reason, we recall now its main features. A $j_{qq}-J_q$-elementary matrix polynomial A of formal degree $n+1$ is called Arov-normalized if $A_{22}(0)>0$, $\tilde{A}_{21}^{[n+1]}(0)>0$ and $A_{21}(0)=0$, whereas a J_q-j_{qq}-elementary matrix polynomial A of formal degree $n+1$ is said to be Arov-normalized if $A_{22}(0)>0$, $\tilde{A}_{12}^{[n+1]}(0)>0$ and $A_{12}(0)=0$. We will use $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$ (respectively, $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$) to denote the set of all Arov-normalized $j_{qq}-J_q$-elementary (respectively, Arov-normalized J_q-j_{qq}-elementary) matrix polynomials of formal degree $n+1$. From [13, Lemma 6, Remarks 14 and 16] we know that the following two statements hold true.

Remark 2.2. $\mathscr{A}_{n+1}^{(j_{qq},J_q)}\subset\mathscr{F}_{n+1}^{(j_{qq},J_q)}$ and $\mathscr{A}_{n+1}^{(J_q,j_{qq})}\subset\mathscr{F}_{n+1}^{(J_q,j_{qq})}$.

Remark 2.3. A belongs to $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$ if and only if $\check{A}\in\mathscr{A}_{n+1}^{(j_{qq},J_q)}$.

In the following we will use the one-to-one correspondence between Arov-normalized $j_{qq}-J_q$-elementary polynomials and nondegenerate $q\times q$-Carathéodory sequences. For the convenience of the reader, we are going to recall some notions and results which are essential for this interrelation. If $(\Gamma_k)_{k=0}^n$ is a sequence of $q\times q$ complex matrices, then let the block Toeplitz matrices $S_{\Gamma,n}$, $U_{\Gamma,n}$ and $T_{\Gamma,n}$ be given by

$$U_{\Gamma,n}:=\begin{pmatrix}\Gamma_0&\Gamma_1^*&\Gamma_2^*&\cdots&\Gamma_n^*\\ \Gamma_1&\Gamma_0&\Gamma_1^*&\cdots&\Gamma_{n-1}\\ \Gamma_2&\Gamma_1&\Gamma_0&\cdots&\Gamma_{n-2}\\ \vdots&\vdots&\vdots&&\vdots\\ \Gamma_n&\Gamma_{n-1}&\Gamma_{n-2}&\cdots&\Gamma_0\end{pmatrix},$$

$$S_{\Gamma,n}:=\begin{pmatrix}\Gamma_0&0&0&\cdots&0&0\\ \Gamma_1&\Gamma_0&0&\cdots&0&0\\ \vdots&\vdots&\vdots&&\vdots&\vdots\\ \Gamma_{n-1}&\Gamma_{n-2}&\Gamma_{n-3}&\cdots&\Gamma_0&0\\ \Gamma_n&\Gamma_{n-1}&\Gamma_{n-2}&\cdots&\Gamma_1&\Gamma_0\end{pmatrix} \tag{2.1}$$

and

$$T_{\Gamma,n} := \operatorname{Re} S_{\Gamma,n}. \tag{2.2}$$

If τ is a nonnegative integer or $\tau = +\infty$, then a sequence $(\Gamma_k)_{k=0}^{\tau}$ of $q \times q$ complex matrices is called $q \times q$ Carathéodory sequence (respectively, nondegenerate $q \times q$ Carathéodory sequence) if $T_{\Gamma,n}$ is nonnegative Hermitian (respectively, positive Hermitian) for every integer n with $0 \leqslant n \leqslant \tau$. The set of all $q \times q$ Carathéodory sequences is intimately connected with the class $\mathscr{C}_q(\mathbb{D})$ of all $q \times q$ Carathéodory functions (in $\mathbb{D}$), i.e., with the set of all matrix-valued functions $\Omega : \mathbb{D} \to \mathbb{C}^{q \times q}$ which are holomorphic in $\mathbb{D}$ and which satisfy $\operatorname{Re} \Omega(z) \geqslant 0$ for all $z \in \mathbb{D}$ (see, e.g., [5,16], [7, Theorems 2.2.1 and 2.2.3], and [11, Part I]). The restriction onto $\mathbb{D}$ of any $j_{qq} - J_q$-elementary polynomial belongs to the Potapov class $\mathscr{P}_{j_{qq},J_q}(\mathbb{D})$. Hence, if $A \in \mathscr{E}_{n+1}^{(j_{qq},J_q)}$, then, for each $z \in \mathbb{D}$, both matrices $A_{12}(z)$ and $A_{22}(z)$ are nonsingular, and the matrix-valued function $(\text{Rstr.}_{\mathbb{D}} A_{12})(\text{Rstr.}_{\mathbb{D}} A_{22})^{-1}$ belongs to $\mathscr{C}_q(\mathbb{D})$ (see [9, Theorem 1]). Thus each $j_{qq} - J_q$-elementary polynomial A of formal degree $n + 1$ is connected to a $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$, namely the sequence of the first $n + 1$ Taylor coefficients of the function $(\text{Rstr.}_{\mathbb{D}} A_{12})(\text{Rstr.}_{\mathbb{D}} A_{22})^{-1}$. This sequence $(\Gamma_k)_{k=0}^{n}$ is called the $q \times q$ Carathéodory sequence associated with the $j_{qq} - J_q$-elementary polynomial A (of formal degree $n + 1$). Similarly, the $q \times q$ Carathéodory sequence $(\Delta_k)_{k=0}^{n}$ associated with a $J_q - j_{qq}$-elementary polynomial of formal degree $n + 1$ is defined (see [13, Section 4]). If $A \in \mathscr{F}_{n+1}^{(j_{qq},J_q)}$ or $A \in \mathscr{F}_{n+1}^{(J_q,j_{qq})}$, then the $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$ associated with A is necessarily nondegenerate (see [13, Lemma 7]). In particular, we see from Remark 2.2 that the $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$ which is associated with an Arov-normalized $j_{qq} - J_q$-elementary polynomial (or Arov-normalized $J_q - j_{qq}$-elementary polynomial) of formal degree $n + 1$ is necessarily nondegenerate. If $A \in \mathscr{A}_{n+1}^{(j_{qq},J_q)}$ and $B \in \mathscr{A}_{n+1}^{(j_{qq},J_q)}$ (respectively, $A \in \mathscr{A}_{n+1}^{(J_q,j_{qq})}$ and $B \in \mathscr{A}_{n+1}^{(J_q,j_{qq})}$) are such that the associated $q \times q$ Carathéodory sequences coincide, then $A = B$ (see [13, Theorem 1]). Moreover, starting from an arbitrary nondegenerate $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$ one can construct explicitly the Arov-normalized $j_{qq} - J_q$-elementary (respectively, Arov-normalized $J_q - j_{qq}$-elementary) polynomial of formal degree $n + 1$ such that $(\Gamma_k)_{k=0}^{n}$ is exactly its associated $q \times q$ Carathéodory sequence: Let $(\Gamma_k)_{k=0}^{n}$ be a nondegenerate $q \times q$ Carathéodory sequence. Using (2.1) it is readily checked that there is a unique sequence $(\gamma_k)_{k=0}^{n}$ of $q \times q$ complex matrices such that $S_{\Gamma,n} \cdot S_{\gamma,n} = I$ (see also [11, Part V, Lemma 25]). This sequence $(\gamma_k)_{k=0}^{n}$ is a nondegenerate $q \times q$ Carathéodory sequence as well (see [11, Part V, Lemma 26]), i.e., the block Toeplitz matrix $T_{\gamma,n}$ is positive Hermitian. The sequence $(\gamma_k)_{k=0}^{n}$ is called the reciprocal $q \times q$ Carathéodory sequence corresponding to $(\Gamma_k)_{k=0}^{n}$. Obviously,

$$\det \Gamma_k \neq 0 \quad \text{and} \quad \gamma_0 = \Gamma_0^{-1}. \tag{2.3}$$

Moreover, one can easily see that $(\gamma_k)_{k=0}^n$ is a nondegenerate $q \times q$ Carathéodory sequence as well and that $(\Gamma_k)_{k=0}^n$ is the reciprocal $q \times q$ Carathéodory sequence corresponding to $(\gamma_k)_{k=0}^n$ (see also [11, Part V, Lemma 26]). For $z \in \mathbb{C}$, let

$$e_{nq}(z) := \left(I_q, zI_q, \ldots, z^n I_q\right), \quad \varepsilon_{nq}(z) := \left(\overline{z}^n I_q, \overline{z}^{n-1} I_q, \ldots, \overline{z} I_q, I_q\right)^*,$$

and

$$\eta_{\Gamma,n}(z) := e_{nq}(z) T_{\Gamma,n}^{-1} e_{nq}^*(0), \quad \rho_{\Gamma,n}(z) := \varepsilon_{nq}^*(0) T_{\Gamma,n}^{-1} \varepsilon_{nq}(z)$$

and

$$\eta_{\gamma,n}(z) := e_{nq}(z) T_{\gamma,n}^{-1} e_{nq}^*(0), \quad \rho_{\gamma,n}(z) := \varepsilon_{nq}^*(0) T_{\gamma,n}^{-1} \varepsilon_{nq}(z).$$

Furthermore, we set $L_{\Gamma,1} := \operatorname{Re} \Gamma_0$, $R_{\Gamma,1} := \operatorname{Re} \Gamma_0$, and, if $n > 0$,

$$z_{\Gamma,n} := (\Gamma_n, \Gamma_{n-1}, \ldots, \Gamma_1), \quad y_{\Gamma,n} := (\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_n^*)^*$$

and

$$L_{\Gamma,n+1} := \operatorname{Re} \Gamma_0 - \frac{1}{4} z_{\Gamma,n} T_{\Gamma,n}^{-1} z_{\Gamma,n}^*, \quad R_{\Gamma,n+1} := \operatorname{Re} \Gamma_0 - \frac{1}{4} y_{\Gamma,n}^* T_{\Gamma,n}^{-1} y_{\Gamma,n}.$$

One can easily see that

$$L_{\gamma,n+1} = \Gamma_0^{-1} L_{\Gamma,n+1} \Gamma_0^{-*} \quad \text{and} \quad R_{\gamma,n+1} = \Gamma_0^{-*} R_{\Gamma,n+1} \Gamma_0^{-1} \tag{2.4}$$

(compare also [11, Part V, Lemma 27]).

Remark 2.4. Since $T_{\Gamma,n}$ is positive Hermitian it is readily checked that all the matrices $L_{\Gamma,n+1}$, $R_{\Gamma,n+1}$, $L_{\gamma,n+1}$ and $R_{\gamma,n+1}$ are positive Hermitian. Moreover $\rho_{\Gamma,n}(0) = L_{\Gamma,n+1}^{-1}$ and $\eta_{\Gamma,n}(0) = R_{\Gamma,n+1}^{-1}$ (see also [11, Part I, Remarks 1 and 2] and [11, Part III, Lemma 11]). The functions $\det \eta_{\Gamma,n}$ and $\det \rho_{\Gamma,r}$ coincide, and there exists a real number r greater than 1 such that $\det \eta_{\Gamma,n}$ and $\det \rho_{\Gamma,n}$ do not vanish in $\{z \in C : |z| \leqslant r\}$ (see, e.g., [6] or [12, Lemma 1]).

We now can state the explicit form of the unique Arov-normalized $j_{qq} - J_q$-elementary polynomial of formal degree $n+1$ the associated Carathéodory sequence of which is $(\Gamma_k)_{k=0}^n$, which is proved in [13, Theorems 1 and 3].

Theorem 2.5. *Let $(\Gamma_k)_{k=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence.*

(a) *There exists a unique Arov-normalized $j_{qq} - J_q$-elementary polynomial $Y_{\Gamma,n}$ of formal degree $n+1$ such that $(\Gamma_k)_{k=0}^n$ is exactly the $q \times q$ Carathéodory sequence associated with $Y_{\Gamma,n}$, namely, $Y_{\Gamma,n} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by*

$$Y_{\Gamma,n}(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} -z\tilde{\rho}_{\gamma,n}^{[n]}(z)\Gamma_0^{-1}\sqrt{L_{\Gamma,n+1}} & \eta_{\gamma,n}(z)\Gamma_0^{-*}\sqrt{R_{\Gamma,n+1}} \\ z\tilde{\rho}_{\Gamma,n}^{[n]}(z)\sqrt{L_{\Gamma,n+1}} & \eta_{\Gamma,n}(z)\sqrt{R_{\Gamma,n+1}} \end{pmatrix}. \tag{2.5}$$

(b) *There exists a unique Arov-normalized $J_q - j_{qq}$-elementary polynomial $Z_{\Gamma,n}$ of formal degree $n+1$ such that $(\Gamma_k)_{k=0}^n$ is exactly the $q \times q$ Carathéodory sequence associated with $Z_{\Gamma,n}$, namely, $Z_{\Gamma,n} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by*

$$Z_{\Gamma,n}(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} -z\sqrt{R_{\Gamma,n+1}}\Gamma_0^{-1}\tilde{\eta}_{\gamma,n}^{[n]}(z) & \sqrt{R_{\Gamma,n+1}}\tilde{\eta}_{\Gamma,n}(z) \\ \sqrt{L_{\Gamma,n+1}}\Gamma_0^{-*}\rho_{\gamma,n}(z) & \sqrt{L_{\Gamma,n+1}}\rho_{\Gamma,n}(z) \end{pmatrix}. \tag{2.6}$$

From the definition of the $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$ associated with $A \in \mathscr{E}_{n+1}^{(j_{qq},J_q)}$, we see immediately that

$$\Gamma_0 = A_{12}(0)(A_{22}(0))^{-1}. \tag{2.7}$$

For functions which belong to the subclass $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$ of $\mathscr{E}_{n+1}^{(j_{qq},J_q)}$ we get a further representation of Γ_0.

Corollary 2.6. *Let $A \in \mathscr{A}_{n+1}^{(j_{qq},J_q)}$, and let $(\Gamma_k)_{k=0}^n$ be the $q \times q$ Carathéodory sequence associated with A. Then*

$$\Gamma_0 = -\left(\tilde{A}_{21}^{[n+1]}(0)\right)^{-1} \tilde{A}_{11}^{[n+1]}(0).$$

Proof. In view of Remark 2.2, the $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$ is nondegenerate. Theorem 2.5 yields $A = Y_{\Gamma,n}$. For each $z \in \mathbb{C}$, we obtain

$$\tilde{A}_{11}^{[n+1]}(z) = -\frac{1}{\sqrt{2}}\sqrt{L_{\Gamma,n+1}}\Gamma_0^{-*}\rho_{\gamma,n}(z)$$

and

$$\tilde{A}_{21}^{[n+1]}(z) = \frac{1}{\sqrt{2}}\sqrt{L_{\Gamma,n+1}}\rho_{\Gamma,n}(z).$$

Thus we get from [11, Part V, Theorem 28] that

$$-\left(\tilde{A}_{21}^{[n+1]}(0)\right)^{-1} \tilde{A}_{11}^{[n+1]}(0) = (\rho_{\Gamma,n}(0))^{-1}\Gamma_0^{-*}\rho_{\gamma,n}(0) = \Gamma_0. \qquad \square$$

We now consider an Arov-normalized $j_{qq} - J_q$-elementary polynomial A of formal degree $n+1$ and its $q \times q$ block partition stated in (1.1). We study the question of uniqueness if one of the four $q \times q$ blocks A_{11}, A_{12}, A_{21} and A_{22} is given. For the block A_{22} this problem was already considered in [13]. Observe that the proof of Theorem 6 in [13] shows that the following result holds true:

Proposition 2.7. *Let A and B belong to $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. If $A_{22} = B_{22}$ and*

$$\operatorname{Im}\left(A_{12}(0)(A_{22}(0))^{-1}\right) = \operatorname{Im}\left(B_{12}(0)(B_{22}(0))^{-1}\right), \tag{2.8}$$

then $A = B$.

The essential step in the proof of Proposition 2.7 is the consideration of the $q \times q$ Carathéodory sequence associated with a given element of $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. Modifying this idea we will be able to give similar results in the cases when one of the other $q \times q$

blocks A_{11}, A_{12} and A_{21} of an Arov-normalized $j_{qq} - J_q$-elementary polynomial A is given. On the other hand, one can easily translate such results into statements on Arov-normalized $J_q - j_{qq}$-elementary polynomials.

Corollary 2.8. *Let A and B belong to* $\mathscr{A}_{n+1}^{(J_q, j_{qq})}$. *If* $A_{22} = B_{22}$ *and*

$$\operatorname{Im}\big((A_{22}(0))^{-1}A_{21}(0)\big) = \operatorname{Im}\big((B_{22}(0))^{-1}B_{21}(0)\big), \tag{2.9}$$

then $A = B$.

Proof. By virtue of Remark 2.3, $\check{A}$ belongs to $\mathscr{A}_{n+1}^{(j_{qq}, J_q)}$. Moreover, we see from the $q \times q$ block partition (1.2) of $\check{A}$ that

$$\operatorname{Im}\big((A_{21}(0))^*(A_{22}(0))^{-*}\big) = -\operatorname{Im}\big((A_{22}(0))^{-1}A_{21}(0)\big).$$

Hence the application of Proposition 2.7 completes the proof. □

Lemma 2.9. *Let* $(\Gamma_k)_{k=0}^n$ *and* $(\Delta_k)_{k=0}^n$ *be nondegenerate* $q \times q$ *Carathéodory sequences, and let* $(\gamma_k)_{k=0}^n$ *and* $(\delta_k)_{k=0}^n$ *be the reciprocal* $q \times q$ *Carathéodory sequences corresponding to* $(\Gamma_k)_{k=0}^n$ *and* $(\Delta_k)_{k=0}^n$, *respectively. Suppose that the matrix-valued functions* $Y_{\Gamma,n}$ *and* $Y_{\Delta,n}$ *given by* (2.5) *satisfy one of the following two conditions*:

(i) *The right upper* $q \times q$ *blocks of* $Y_{\Gamma,n}$ *and* $Y_{\Delta,n}$ *coincide.*

(ii) *The left upper* $q \times q$ *blocks of* $Y_{\Gamma,n}$ *and* $Y_{\Delta,n}$ *coincide.*

Then $\gamma_k = \delta_k$ *for each integer k which satisfies* $1 \leqslant k \leqslant n$, *and there is a Hermitian matrix* $H \in \mathbb{C}^{q \times q}$ *such that* $\gamma_0 = \delta_0 + \mathrm{i}H$.

Proof. First assume that (i) is satisfied. Then

$$\eta_{\gamma,n}(z)\Gamma_0^{-*}\sqrt{R_{\Gamma,n+1}} = \eta_{\delta,n}(z)\Delta_0^{-*}\sqrt{R_{\Delta,n+1}} \tag{2.10}$$

for each $z \in \mathbb{C}$. From (2.8) we can conclude

$$\eta_{\gamma,n}(0)\Gamma_0^{-*}\sqrt{R_{\Gamma,n+1}} = R_{\gamma,n+1}^{-1}\Gamma_0^{-*}\sqrt{R_{\Gamma,n+1}} = \Gamma_0\,\sqrt{R_{\Gamma,n+1}}^{-1}$$

and

$$\eta_{\delta,n}(0)\Delta_0^{-*}\sqrt{R_{\Delta,n+1}} = \Delta_0\sqrt{R_{\Delta,n+1}}^{-1}.$$

In view of (2.10) this implies

$$\Gamma_0^{-*}\sqrt{R_{\Gamma,n+1}} = \Delta_0^{-*}\sqrt{R_{\Gamma,n+1}},$$

and hence $\eta_{\gamma,n} = \eta_{\delta,n}$. Analogously, $\rho_{\gamma,n} = \rho_{\delta,n}$ follows from (ii). Applying [11, Part III, Theorem 16 and Remark 18] we get $T_{\Gamma,n} = T_{\Delta,n}$. Thus the assertion is verified. □

It should be mentioned that the identity $T_{\Gamma,n} = T_{\Delta,n}$ can also be derived using results due to [17] or [15].

Proposition 2.10. *Let A and B belong to* $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. *If* $A_{12} = B_{12}$ *and*

$$\operatorname{Im}\big(A_{12}(0)(A_{22}(0))^{-1}\big) = \operatorname{Im}\big(B_{12}(0)(B_{22}(0))^{-1}\big), \tag{2.11}$$

then $A = B$.

Proof. Let $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$ be the $q \times q$ Carathéodory sequences associated with A and B, respectively, and let $(\gamma_k)_{k=0}^n$ and $(\delta_k)_{k=0}^n$ be the reciprocal $q \times q$ Carathéodory sequences corresponding to $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$, respectively. Assume that $A_{12} = B_{12}$. Using Theorem 2.5 and Lemma 2.9 we get $\gamma_k = \delta_k$ for each $k \in \{1, 2, \ldots, n\}$ and $\gamma_0 = \delta_0 + \mathrm{i}H$ with some Hermitian $q \times q$ complex matrix H. From (2.3) it follows that $\Gamma_0^{-1} = \Delta_0^{-1} + \mathrm{i}H$. We know that (2.7) and, analogously, $\Delta_0 = B_{12}(0)(B_{22}(0))^{-1}$ are fulfilled. Supposing (2.11) we get from (2.3) that

$$\begin{aligned}
\operatorname{Im}\delta_0 &= \operatorname{Im}\big(\Delta_0^{-1}\big)\\
&= \operatorname{Im}\big([B_{12}(0)(B_{22}(0))^{-1}]^{-1}\big)\\
&= \operatorname{Im}\big([A_{12}(0)(A_{22}(0))^{-1}]^{-1}\big)\\
&= \operatorname{Im}\big(\Gamma_0^{-1}\big)\\
&= \operatorname{Im}\gamma_0\\
&= \operatorname{Im}\big(\delta_0 + \mathrm{i}H\big)\\
&= \operatorname{Im}\delta_0 + H.
\end{aligned}$$

Thus $H = 0$. Then we get $\gamma_k = \delta_k$ and hence $\Gamma_k = \Delta_k$ for each all $k \in \{0, 1, \ldots, n\}$. The application of Theorem 2.5 completes the proof. □

Corollary 2.11. *Let A and B belong to* $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$. *If* $A_{21} = B_{21}$ *and* (2.9) *are satisfied, then* $A = B$.

Proof. Use Remark 2.3 and Proposition 2.10. □

Proposition 2.12. *Let A and B belong to* $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. *If* $A_{11} = B_{11}$ *and*

$$\operatorname{Im}\left(\left(\tilde{A}_{21}^{[n+1]}(0)\right)^{-1}\tilde{A}_{11}^{[n+1]}(0)\right) = \operatorname{Im}\left(\left(\tilde{B}_{21}^{[n+1]}(0)\right)^{-1}\tilde{B}_{11}^{[n+1]}(0)\right), \tag{2.12}$$

then $A = B$.

Proof. Using Lemma 2.9, Corollary 2.6 and Theorem 2.5, Proposition 2.12 can be proved analogously to Proposition 2.10. We omit the details. □

Remark 2.13. Let B belong to $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$. Setting $b_{rs} := \check{B}_{rs}$ and $c_{rs} := \tilde{B}_{rs}^{[n+1]}$ one can easily see that $\tilde{b}_{rs}^{[n+1]} = \check{c}_{rs}$ and hence $\tilde{b}_{rs}^{[n+1]}(0) = (c_{rs}(0))^*$ for every choice of $r \in \{1, 2\}$ and $s \in \{1, 2\}$. Moreover,

$$\operatorname{Im}\left(\tilde{B}_{11}^{[n+1]}(0)\left(\tilde{B}_{12}^{[n+1]}(0)\right)^{-1}\right) = -\operatorname{Im}\big((\tilde{b}_{12}^{[n+1]}(0))^{-1}\tilde{b}_{11}^{[n+1]}(0)\big).$$

Corollary 2.14. *Let A and B belong to $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$. If $A_{11} = B_{11}$ and*

$$\operatorname{Im}\left(\tilde{A}_{11}^{[n+1]}(0)\left(\tilde{A}_{12}^{[n+1]}(0)\right)^{-1}\right) = \operatorname{Im}\left(\tilde{B}_{11}^{[n+1]}(0)\left(\tilde{B}_{12}^{[n+1]}(0)\right)^{-1}\right) \tag{2.13}$$

are satisfied, then $A = B$.

Proof. Remark 2.3 shows that $\check{A}$ and $\check{B}$ belong to $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. In view of (1.2) the assertion follows immediately from Remark 2.13 and Proposition 2.12. □

Lemma 2.15. *Let $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$ be nondegenerate $q \times q$ Carathéodory sequences. Suppose that the matrix-valued functions $Y_{\Gamma,n}$ and $Y_{\Delta,n}$ given by* (2.5) *satisfy one of the following two conditions*:
(i) *The left lower $q \times q$ blocks of $Y_{\Gamma,n}$ and $Y_{\Delta,n}$ coincide.*
(ii) *The right lower $q \times q$ blocks of $Y_{\Gamma,n}$ and $Y_{\Delta,n}$ coincide.*
Then $\Gamma_k = \Delta_k$ for all $k \in \{1, 2, \ldots, n\}$ and $\Gamma_0 = \Delta_0 + \mathrm{i}H$ with some Hermitian $q \times q$ complex matrix H.

Proof. First assume that (i) holds. For all $z \in \mathbb{C}$, we see from (2.5) that

$$\frac{1}{\sqrt{2}} z\tilde{\rho}_{\Gamma,n}^{[n]}(z)\sqrt{L_{\Gamma,n+1}} = \frac{1}{\sqrt{2}} z\tilde{\rho}_{\Delta,n}^{[n]}(z)\sqrt{L_{\Delta,n+1}}$$

and, in view of Remark 2.4,

$$\sqrt{\rho_{\Gamma,n}(0)}^{-1}\rho_{\Gamma,n}(z) = \sqrt{\rho_{\Delta,n}(0)}^{-1}\rho_{\Delta,n}(z).$$

In particular, $\sqrt{\rho_{\Gamma,n}(0)} = \sqrt{\rho_{\Delta,n}(0)}$. Thus $\rho_{\Gamma,n} = \rho_{\Delta,n}$ follows. Analogously, $\eta_{\Gamma,n} = \eta_{\Delta,n}$ can be obtained from (ii). (This implication is already verified in [13, Lemma 9]. It was used to prove Proposition 2.7.) Applying [11, Part III, Remark 18 and Theorem 16] we see that $T_{\Gamma,n} = T_{\Delta,n}$. Consequently, we get $\Gamma_k = \Delta_k$ for all $k \in \{1, 2, \ldots, n\}$ and $\operatorname{Re}\Gamma_0 = \operatorname{Re}\Delta_0$. Hence the proof is complete. □

Using Lemma 2.15, Theorem 2.5, and Corollary 2.7 one can now easily check the following result concerning the equality of two Arov-normalized $j_{qq} - J_q$-elementary polynomials. Since the considerations are similar to those of Lemma 2.9, we will omit the proof.

Proposition 2.16. *Let A and B belong to $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. If $A_{21} = B_{21}$ and* (2.12) *are satisfied, then $A = B$.*

Analogous to the proof of Corollary 2.14 one can check the following.

Corollary 2.17. *Let A and B belong to* $\mathscr{A}_{n+1}^{(J_q,j_{qq})}$. *If* $A_{12} = B_{12}$ *and* (2.13) *are satisfied, then* $A = B$.

3. Completion problems for Arov-normalized $j_{qq} - J_q$-elementary factors

First we are going to give an answer to the following completion problem.

(A 22): Let P be a $q \times q$ matrix polynomial, and let m be a positive integer. Describe the set $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{22}$ of all Arov-normalized $j_{qq} - J_q$-elementary polynomials of formal degree m such that $A_{22} = P$. In particular, characterize the case $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{22} \neq \emptyset$.

Theorem 3.1. *Let P be a* $q \times q$ *matrix polynomial of formal degree n.*

(a) *The following statements are equivalent*:

- (i) $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{22} \neq \emptyset$.
- (ii) $\mathscr{A}_m^{(J_q,j_{qq})}[P]_{22} \neq \emptyset$.
- (iii) *The following three conditions are satisfied*:
  - (α) $m > n$.
  - (β) *The function* det P *nowhere vanishes in* $\mathbb{D} \cup \mathbb{T}$.
  - (γ) *The matrix* $P(0)$ *is positive Hermitian.*

(b) *Let conditions* (α), (β) *and* (γ) *be satisfied. For every Hermitian* $q \times q$ *complex matrix* H, *then there are a unique function* $A \in \mathscr{A}_m^{(j_{qq},J_q)}[P]_{22}$ *which satisfies*

$$\operatorname{Im}\big(A_{12}(0)(A_{22}(0)^{-1})\big) = H. \tag{3.1}$$

and a unique $B \in \mathscr{A}_m^{(J_q,j_{qq})}[P]_{22}$ *which fulfills*

$$\operatorname{Im}\big((B_{22}(0))^{-1}(B_{21}(0))\big) = H. \tag{3.2}$$

Proof. From [13, Lemma 3] one can easily see that (β) is necessary for (i) and necessary for (ii). If (i) or (ii) holds true, then (γ) follows from the definition of Arov-normalization. If (i) and $A \in \mathscr{A}_m^{(j_{qq},J_q)}[P]_{22}$ (respectively, (ii) and $A \in \mathscr{A}_m^{(J_q,j_{qq})}[P]_{22}$) are fulfilled, then Theorem 2.5 shows that $P = \eta_{\Gamma,m-1}\sqrt{R_{\Gamma,m}}$ (respectively, $P = \sqrt{L_{\Gamma,m}}\rho_{\Gamma,m-1}$), where $(\Gamma_k)_{k=0}^{m-1}$ is the $q \times q$ Carathéodory sequence associated with A. Consequently, (α) is a consequence of (i) (respectively, (ii)). It remains to verify that (b) holds true. Suppose that (α), (β) and (γ) are satisfied and that H is a Hermitian $q \times q$ complex matrix. The case $m = n + 1$ is already proved in [13, Theorem 6]. Thus we can assume that $r := m - (n + 1)$ is positive, and that A belongs to $\mathscr{A}_{n+1}^{(j_{qq},J_q)}[P]_{22}$ and satisfies (3.1). The matrix-valued function $D_r : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ defined by $D_r(z) := \operatorname{diag}(z^r I_q, I_q)$ obviously belongs to $\mathscr{E}_r^{(j_{qq},j_{qq})}$. Then we obtain from [13, Remark 12] that $A_r := AD_r$ belongs to $\mathscr{E}_{n+1+r}^{(j_{qq},J_q)}$. If $A_r = (A_{r;jk})_{j,k=1}^2$ is the $q \times q$ block partition of A_r, then $A_{r;12} = A_{12}$, $A_{r;22} = A_{22} = P$ and $A_{r;21}(z) =$

$z^r A_{21}(z)$ for all $z \in \mathbb{C}$, which implies $A_{r;21}(0) = 0$, $\tilde{A}_{r;21}^{[n+1+r]} = \tilde{A}_{21}^{[n+1]}$ and, in particular, $\tilde{A}_{r;21}^{[n+1+r]}(0) = \tilde{A}_{21}^{[n+1]}(0) > 0$. Hence $A_r \in \mathscr{A}_{n+1+r}^{(j_{qq},J_q)}[P]_{22}$ and $\operatorname{Im}(A_{r;12}(0)(P(0))^{-1}) = H$. The uniqueness of a function $A \in \mathscr{A}_{n+1+r}^{(j_{qq},J_q)}[P]$ which satisfies (3.1) follows immediately from Proposition 2.7. Analogously, one can see that there is a unique $B \in \mathscr{A}_{n+1+r}^{(J_q,j_{qq})}[P]_{22}$ which satisfies (3.2). $\square$

Using [12, Theorem 4] the proof of Theorem 6 in [13] shows how one can construct explicitly the unique function $A \in \mathscr{A}_{n+1+r}^{(j_{qq},J_q)}[P]_{22}$ which fulfills (3.1). Thus $A_r = AD_r$ also admits an explicit construction. Similarly, the unique function $B \in \mathscr{A}_{n+1+r}^{(J_q,j_{qq})}[P]_{22}$ which satisfies (3.2) can be constructed explicitly. We now turn our attention to the situation that the left lower $q \times q$ block is given, i.e., we study the following completion problem:

(A 21): Let P be a $q \times q$ matrix polynomial, and let m be a positive integer. Describe the set $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{21}$ of all Arov-normalized $j_{qq} - J_q$-elementary polynomials of formal degree m such that $A_{21} = P$. In particular, characterize the situation $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{21} \neq \emptyset$.

Lemma 3.2. *Let H be a Hermitian $q \times q$ complex matrix.*

(a) *Let P be a $q \times q$ matrix polynomial of formal degree not greater than n such that* $\det P$ *nowhere vanishes in $\mathbb{D} \cup \mathbb{T}$ and that $P(0)$ is a positive Hermitian matrix. Then there is a unique nondegenerate $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^n$ such that $\eta_{\Gamma,n} = P$ and* $\operatorname{Im} \Gamma_0 = H$.

(b) *Let Q be a $q \times q$ matrix polynomial of formal degree not greater than n such that* $\det Q$ *nowhere vanishes in $\mathbb{D} \cup \mathbb{T}$ and that $Q(0)$ is a positive Hermitian matrix. Then there is a unique nondegenerate $q \times q$ Carathéodory sequence $(\Delta_k)_{k=0}^n$ such that $\rho_{\Delta,n} = Q$ and* $\operatorname{Im} \Delta_0 = H$.

(c) *The $q \times q$ Carathéodory sequences $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$ coincide if and only if*

$$(P(z))^{-*} P(0) (P(z))^{-1} = (Q(z))^{-1} Q(0) (Q(z))^{-*} \tag{3.3}$$

for every choice of z in $\mathbb{T}$.

Proof. Let $(a_k)_{k=0}^n$ and $(b_k)_{k=0}^n$ be the unique sequences of $q \times q$ complex matrices such that $P(z) = \sum_{k=0}^n a_k z^k$ and $Q(z) = \sum_{k=0}^n b_k z^k$ for all $z \in \mathbb{C}$. In view of (2.1) and [12, Theorems 5 and 6] we know that there are unique positive definite sequences $(C_k)_{k=0}^n$ and $(D_k)_{k=0}^n$ of $q \times q$ complex matrices such that $(U_{C,n})^{-1} e_{nq}^*(0) = (a_0^*, a_1^*, \dots, a_n^*)^*$ and $\varepsilon_{nq}^*(0)(U_{D,n})^{-1} = (b_n, b_{n-1}, \dots, b_0)$, where [12, Theorem 9] shows that the sequences $(C_k)_{k=0}^n$ and $(D_k)_{k=0}^n$ coincide if and only if (3.3) holds for all $z \in \mathbb{T}$. Hence $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$, given by $\Gamma_0 := C_0 + \mathrm{i}H$, $\Delta_0 := D_0 + \mathrm{i}H$ and $\Gamma_k := 2C_k$, $\Delta_k := 2D_k$ for all $k \in \{1, 2, \dots, n\}$, are nondegenerate $q \times q$ Carathéodory sequences which fulfill $T_{\Gamma,n}^{-1} e_{nq}^*(0) = (a_0^*, a_1^*, \dots, a_n^*)^*$ and $\varepsilon_{nq}^*(0) T_{\Delta,n}^{-1} =$

$(b_n, b_{n-1}, \dots, b_0)$, and the sequences $(\Gamma_k)_{k=0}^n$ and $(\Delta_k)_{k=0}^n$ coincide if and only if (3.3) is valid for each $z \in \mathbb{T}$. If $(\Gamma_k^{\#})_{k=0}^n$ (respectively, $(\Delta_k^{\#})_{k=0}^n$) is an arbitrary nondegenerate Carathéodory sequence such that $\eta_{\Gamma^{\#},n} = P$ and $\operatorname{Im} \Gamma_0^{\#} = H$ (respectively, $\rho_{\Delta^{\#},n} = Q$ and $\operatorname{Im} \Delta_0^{\#} = H$), then $T_{\Gamma^{\#},n}^{-1} e_{nq}^*(0) = (a_0^*, a_1^*, \dots, a_n^*)^*$ (respectively, $\varepsilon_{nq}^*(0) T_{\Delta^{\#},n}^{-1} = (b_n, b_{n-1}, \dots, b_0)$), i.e., then $(C_k^{\#})_{k=0}^n$ defined by $C_0^{\#} := \operatorname{Re} \Gamma_0$ and $C_k^{\#} := \frac{1}{2}\Gamma_k$ for each $k \in \{1, 2, \dots, n\}$ (respectively, $(\Delta_k^{\#})_{k=0}^n$ defined by $\Delta_0^{\#} := \operatorname{Re} D_0$ and $\Delta_k^{\#} := \frac{1}{2} D_k$ for each $k \in \{1, 2, \dots, n\}$) is a positive definite sequence of complex $q \times q$ matrices such that $(U_{C^{\#},n})^{-1} e_{nq}^*(0) = (a_0^*, a_1^*, \dots, a_n^*)^*$ (respectively, $\varepsilon_{nq}^*(0)(U_{D^{\#},n})^{-1} = (b_n, b_{n-1}, \dots, b_0)$). Thus the sequences $(C_k^{\#})_{k=0}^n$ and $(C_k)_{k=0}^n$ (respectively, the sequences $(D_k^{\#})_{k=0}^n$ and $(D_k)_{k=0}^n$) coincide. Hence $\Gamma_k^{\#} = \Gamma_k$ (respectively, $\Delta_k^{\#} = \Delta_k$) follows for all $k \in \{0, 1, \dots, n\}$. □

Theorem 3.3. *Let P be a $q \times q$ matrix polynomial of formal degree n.*

(a) *The following statements are equivalent*:

- (i) $\mathscr{A}_m^{(j_{qq}, J_q)}[P]_{21} \neq \emptyset$.
- (ii) $\mathscr{A}_m^{(J_q, j_{qq})}[P]_{12} \neq \emptyset$.
- (iii) *The following four conditions are satisfied*:
  - (α) $m = n$.
  - (β) *The function* $\det \tilde{P}^{[n]}$ *nowhere vanishes in* $\mathbb{D} \cup \mathbb{T}$.
  - (γ) *The matrix* $\tilde{P}^{[n]}(0)$ *is positive Hermitian.*
  - (δ) $P(0) = 0$.

(b) *Let conditions* (α), (β), (γ) *and* (δ) *be fulfilled. For every $q \times q$ Hermitian complex matrix H, there are a unique function $A \in \mathscr{A}_m^{(j_{qq}, J_q)}[P]_{21}$ which satisfies*

$$\operatorname{Im}\left(-\left(\tilde{A}_{21}^{[n]}(0)\right)^{-1} \tilde{A}_{11}^{[n]}(0)\right) = H \tag{3.4}$$

and a unique function $B \in \mathscr{A}_m^{(J_q, j_{qq})}[P]_{12}$ which fulfills

$$\operatorname{Im}\left(\tilde{B}_{11}^{[n]}(0)\left(\tilde{B}_{12}^{[n]}(0)\right)^{-1}\right) = H. \tag{3.5}$$

Proof. First we verify that (iii) is necessary for (i). Assume that there is a function A which belongs to $\mathscr{A}_m^{(j_{qq}, J_q)}[P]_{21}$. Since A is Arov-normalized we see that $\tilde{P}^{[m]}(0)$ is positive Hermitian and that (δ) holds. Denote $(\Gamma_k)_{k=0}^{m-1}$ the $q \times q$ Carathéodory sequence associated with A. According to Theorem 2.5 the function A admits the representation $A = Y_{\Gamma,m-1}$. For each $z \in \mathbb{C}$, we get

$$P(z) = A_{21}(z) = \frac{1}{\sqrt{2}} z \tilde{\rho}_{\Gamma,m-1}^{[m-1]}(z) \sqrt{L_{\Gamma,m}} \tag{3.6}$$

and, therefore, $m \leqslant n$ and

$$\tilde{P}^{[m]}(z) = \frac{1}{\sqrt{2}} \sqrt{L_{\Gamma,m}} \rho_{\Gamma,m-1}(z).$$

Hence Remark 2.4 implies that $\det \tilde{P}^{(m)}$ nowhere vanishes in $\mathbb{D} \cup \mathbb{T}$, and that

$$\tilde{P}^{[m]}(0) = \frac{1}{\sqrt{2}}\sqrt{L_{\Gamma,m}}\rho_{\Gamma,m-1}(0) = \frac{1}{\sqrt{2}}\sqrt{L_{\Gamma,m}}^{-1} \neq 0.$$

Since P is a $q \times q$ matrix polynomial of formal degree n we obtain (α). Thus we have proved that (iii) is necessary for (i).

Conversely, we assume now that (iii) holds. In particular, $n \geqslant 1$. Let $(a_k)_{k=0}^{n}$ be the unique sequence of $q \times q$ complex matrices such that $P(z) = \sum_{k=0}^{n} a_k z^k$ for all $z \in \mathbb{C}$. Because of (δ) we have $a_0 = 0$. Hence the $q \times q$ matrix polynomial Q given by $Q(z) := \sum_{k=0}^{n-1} a_{k+1} z^k$ satisfies $P(z) = zQ(z)$ for all $z \in \mathbb{C}$ and $\tilde{P}^{[n]} = \tilde{Q}^{[n-1]}$. In particular, conditions (β) and (γ) show that $\det \tilde{Q}^{[n-1]}$ has no zeros in $\mathbb{D} \cup \mathbb{T}$, and that the matrix $\tilde{Q}^{[n-1]}(0)$ is positive Hermitian. Thus $R := 2\tilde{Q}^{[n-1]}(0)\tilde{Q}^{[n-1]}$ is a $q \times q$ matrix polynomial of formal degree not greater than $n-1$ for which the function $\det R$ nowhere vanishes in $\mathbb{D} \cup \mathbb{T}$, and $R(0) = 2(\tilde{Q}^{[n-1]}(0))^2 > O_{q\times q}$. Moreover, we have $\tilde{Q}^{[n-1]} = \frac{1}{\sqrt{2}}\sqrt{R(0)}^{-1} R$. Let H be a Hermitian $q \times q$ complex matrix. By virtue of Lemma 3.2, there is a (unique) nondegenerate $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n-1}$ such that $\operatorname{Im} \Gamma_0 = H$ and $\rho_{\Gamma,n-1} = R$. In particular, we have $\rho_{\Gamma,n-1}(0) = R(0) = 2(\tilde{Q}^{[n-1]}(0))^2$, and hence, in view of Remark 2.4,

$$\begin{aligned}
\tilde{\rho}_{\Gamma,n-1}^{[n-1]} &= \tilde{R}^{[n-1]} \\
&= 2Q\big[\tilde{Q}^{[n-1]}(0)\big] \\
&= \sqrt{2}Q\sqrt{\rho_{\Gamma,n-1}(0)} \\
&= \sqrt{2}Q\sqrt{L_{\Gamma,n}}^{-1}.
\end{aligned} \tag{3.7}$$

From Theorem 2.5 we know that $Y_{\Gamma,n-1}$ given by (2.5) belongs to $\mathscr{A}_n^{(j_{qq},J_q)}$ and that the associated $q \times q$ Carathéodory sequence of which is $(\Gamma_k)_{k=0}^{n-1}$. Thus we get from (3.7) that the left lower $q \times q$ block A_{21} of $A := Y_{\Gamma,n-1}$ satisfies

$$A_{21}(z) = \frac{1}{\sqrt{2}} z \tilde{\rho}_{\Gamma,n-1}^{[n-1]}(z)\sqrt{L_{\Gamma,n}} = zQ(z) = P(z)$$

for all $z \in \mathbb{C}$. Therefore A belongs to $\mathscr{A}_m^{(j_{qq},J_q)}[P]_{21}$. Since $\operatorname{Im} \Gamma_0 = H$ is valid, the application of Corollary 2.6 yields (3.4). In particular, (i) holds true. Thus we have verified that (i) and (iii) are equivalent. Further, we see from Proposition 2.16 that there is only one function $A \in \mathscr{A}_m^{(j_{qq},J_q)}$ which satisfies (3.4).

We now verify the equivalence of (i) and (ii). Setting $Q := \check{P}$ and $R := \tilde{P}^{[n]}$ we have $\tilde{Q}^{[n]} = \check{R}$. Hence $\det \tilde{Q}^{[n]}(z) = \overline{\det \tilde{P}^{[n]}(\overline{z})}$ for all $z \in \mathbb{C}$, $Q(0) = (P(0))^*$ and $\tilde{Q}^{[n]}(0) = (\tilde{P}^{[n]}(0))^*$. Thus we see that conditions (β), (γ) and (δ) are satisfied if and only if the following three conditions are satisfied:

(β′) The function $\det \tilde{Q}^{[n]}$ nowhere vanishes in $\mathbb{D}$.
(γ′) The matrix $\tilde{Q}^{[n]}(0)$ is positive Hermitian.
(δ′) $Q(0) = 0$.

First we now suppose that (i) is fulfilled, i.e., that (α), (β), (γ) and (δ) hold. Let $Q := \check{P}$. Then (α), (β′), (γ′) and (δ′) are valid. Let $H \in \mathbb{C}^{q\times q}$ be Hermitian. Then we already know that there is a unique $A \in \mathscr{A}_m^{(j_{qq},J_q)}[Q]_{21}$ such that

$$\operatorname{Im}\left(-\left(\tilde{A}_{21}^{[n]}(0)\right)^{-1}\tilde{A}_{11}^{[n]}(0)\right) = -H. \tag{3.8}$$

According to Remark 2.3 and (1.2), $B := \check{A}$ belongs to $\mathscr{A}_m^{(J_q,j_{qq})}[P]_{12}$, and we see from Remark 2.13, (1.2) and (3.8) that (3.5) holds. In particular, (ii) is satisfied. From Corollary 2.17 we see that a function $B \in \mathscr{A}_m^{(J_q,j_{qq})}[P]_{12}$ which satisfies (3.5) is uniquely determined. Conversely, we now suppose that (ii) holds. Let $B \in \mathscr{A}_m^{(J_q,j_{qq})}[P]_{12}$. We set again $Q := \check{P}$. Remark 2.3 and (1.2) provide that $A := \check{B}$ belongs to $\mathscr{A}_m^{(j_{qq},J_q)}[Q]_{21}$. Since (iii) is necessary for (i) it follows that (α), (β′), (γ′) and (δ′) are satisfied. Then we know that (α), (β), (γ) and (δ) hold as well. Since (iii) is sufficient for (i), we get finally (i). □

We now study the case that the last $q \times 2q$ block row of an Arov-normalized $j_{qq} - J_q$-elementary polynomial is prescribed. More precicely, we discuss the following completion problem:

Let P_{21} and P_{22} be $q \times q$ matrix polynomials, and let m be a positive integer. Describe that set $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet}$ of all Arov-normalized $j_{qq} - J_q$-elementary polynomials of formal degree m such that $A_{21} = P_{21}$ and $A_{22} = P_{22}$. In particular, characterize the situation $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet} \neq \emptyset$.

An answer to this question is given by the following theorem.

Theorem 3.4. *Let P_{21} be a $q \times q$ matrix polynomial of formal degree n_1, and let P_{22} be a $q \times q$ matrix polynomial of formal degree n_2. Then:*

(a) *The following statements are equivalent:*

  (i) $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet} \neq \emptyset$.

  (ii) *The following conditions are satisfied:*

    (α) $n_2 < n_1$ *and* $m = n_1$.

    (β) *The functions* $\det \tilde{P}_{21}^{[n_1]}$ *and* $\det P_{22}$ *do not vanish in* $\mathbb{D} \cup \mathbb{T}$.

    (γ) *Both matrices* $\tilde{P}_{21}^{[n]}(0)$ *and* $P_{22}(0)$ *are positive Hermitian.*

    (δ) $P_{21}(0) = 0$.

    (η) *For all* $z \in \mathbb{T}$,

$$P_{22}(z)(P_{22}(z))^* = P_{21}(z)(P_{21}(z))^*. \tag{3.9}$$

(b) *Let conditions (α), (β), (γ), (δ) and (η) be fulfilled. For every Hermitian $q \times q$ complex matrix H, there is a unique function $A \in \mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet}$ such that* $\operatorname{Im}(A_{12}(0)(A_{22}(0))^{-1}) = H$.

Proof. Obviously,

$$\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet} = \left(\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}]_{21}\right) \cap \left(\mathscr{A}_m^{(j_{qq},J_q)}[P_{22}]_{22}\right).$$

First suppose that (i) holds. We know from Theorems 3.1 and 3.3 that $n_2 < m = n_1$, (β), (γ) and (δ) are satisfied. Assume that A belongs to $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet}$. Then $\mathrm{Rstr.}_{\mathbb{D}} A$ is a $j_{qq} - J_q$-inner function. Thus it follows that $(\mathrm{Rstr.}_{\mathbb{D}} A_{22})^{-1}\mathrm{Rstr.}_{\mathbb{D}} A_{21}$ is an inner function which belongs to the Schur class $\mathscr{S}_{q\times q}(\mathbb{D})$ (see [9, Lemma 8(a) and (b)]). Hence we get from (β) and a continuity argument that

$$\begin{aligned} I_q &= (A_{22}(z))^{-1}A_{21}(z)^* A_{22}(z)A_{21}(z) \\ &= ((P_{22}(z))^{-1}P_{21}(z))^*(P_{22}(z))^{-1}P_{21}(z) \\ &= P_{21}^*(z)(P_{22}(z))^{-*}(P_{22}(z))^{-1}P_{21}(z) \end{aligned}$$

for all $z \in \mathbb{T}$. Hence (η) follows. It remains to verify part (b). Suppose that (α), (β), (γ), (δ) and (η) are fulfilled, and let $H \in \mathbb{C}^{q\times q}$ be Hermitian. Let $n := n_1 - 1$ and let $(a_k)_{k=0}^{n+1}$ be the unique sequence of $q \times q$ complex matrices such that $P_{21}(z) := \sum_{k=0}^{n+1} a_k z^k$ for all $z \in \mathbb{C}$. In view of (δ) we have $a_0 = 0$, and the $q \times q$ matrix polynomial R given by $R(z) := \sum_{k=0}^{n} a_{k+1} z^k$ for all $z \in \mathbb{C}$ satisfies $\tilde{P}_{21}^{[n+1]} = \tilde{R}^{[n]}$. Using (η) and (β), for each $z \in \mathbb{T}$, we get

$$(\tilde{R}^{[n]}(z))^*\tilde{R}^{[n]}(z) = |z|^{2(n+1)}P_{21}(z)(P_{21}(z))^* = P_{22}(z)(P_{22}(z))^*$$

and hence

$$(\tilde{R}^{[n]}(z))^{-1}(\tilde{R}^{[n]}(z))^{-*} = (P_{22}(z))^{-*}(P_{22}(z))^{-1}. \tag{3.10}$$

From (α) and (γ) we can conclude that $P := 2P_{22}P_{22}(0)$ and $Q := 2\tilde{R}^{[n]}(0)\tilde{R}^{[n]}$ are $q \times q$ matrix polynomials of formal degree not greater than n with $P(0) = (\sqrt{2}P_{22}(0))^2 > 0_{q\times q}$ and $Q(0) = (\sqrt{2}\tilde{R}^{[n]}(0))^2 = (\sqrt{2}\tilde{P}_{21}^{[n+1]}(0))^2 > 0_{q\times q}$. Moreover we see from (β) that none of the functions $\det P$ and $\det Q$ vanishes in $\mathbb{D} \cup \mathbb{T}$. From (β) and (3.10) it follows that

$$\begin{aligned} (P(z))^{-*}P(0)(P(z))^{-1} &= \frac{1}{2}(P_{22}(z))^{-*}(P_{22}(z))^{-1} \\ &= (Q(z))^{-1}Q(0)(Q(z))^{-*} \end{aligned}$$

is satisfied for each $z \in \mathbb{T}$. Applying Lemma 3.2 we obtain that there is a unique nondegenerate $q \times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^{n}$ such that $\eta_{\Gamma,n} = P$ and $\rho_{\Gamma,n} = Q$. According to Theorem 2.5 the function $Y_{\Gamma,n}$ given by (2.5) belongs to $\mathscr{A}_{n+1}^{(j_{qq},J_q)}$. In view of $n + 1 = n_1 = m$, $Y_{\Gamma,n} \in \mathscr{A}_m^{(j_{qq},J_q)}$ follows. From Remark 2.4 and condition (γ) we see

$$\sqrt{L_{\Gamma,n+1}} = \sqrt{\rho_{\Gamma,n}(0)}^{-1} = \sqrt{Q(0)}^{-1} = \left(\sqrt{2}\tilde{P}_{21}^{[n+1]}(0)\right)^{-1}$$

and

$$\sqrt{R_{\Gamma,n+1}} = \sqrt{\eta_{\Gamma,n}(0)}^{-1} = \sqrt{P(0)}^{-1} = \left(\sqrt{2}P_{22}(0)\right)^{-1}.$$

Then we can conclude that

$$\frac{1}{\sqrt{2}}\eta_{\Gamma,n}\sqrt{R_{\Gamma,n+1}} = \frac{1}{2}P(P_{22}(0))^{-1} = P_{22},$$

i.e., the right lower $q \times q$ block of $Y_{\Gamma,n}$ coincides with P_{22}. Further, we have

$$\tilde{Q}^{[n]} = 2R(\tilde{R}^{[n]}(0))^* = 2R(\tilde{P}_{21}^{[n+1]}(0))^* = 2R\tilde{P}_{21}^{[n+1]}(0)$$

and

$$\frac{1}{\sqrt{2}}z\tilde{\rho}_{\Gamma,n}^{[n]}\sqrt{L_{\Gamma,n+1}} = \frac{1}{2}z\tilde{Q}^{[n]}(z)(\tilde{P}_{21}^{[n+1]}(0))^{-1} = zR(z) = P_{21}(z)$$

for each $z \in \mathbb{C}$, i.e., the left lower $q \times q$ block of $Y_{\Gamma,n}$ coincides with P_{21} as well. Hence $Y_{\Gamma,n}$ belongs to $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet}$. It remains to check the uniqueness. However, this is a consequence of (α), (β), (γ), the inclusion $\mathscr{A}_m^{(j_{qq},J_q)}[P_{21}, P_{22}]_{2\bullet} \subseteq \mathscr{A}_m^{(j_{qq},J_q)}[P_{22}]_{22}$ and Theorem 3.1. □

Similar to the completion problems studied above we obtain an answer to the following question:

Let P_{12} and P_{22} be $q \times q$ matrix polynomials, and let m be a positive integer. Describe the set $\mathscr{A}_m^{(J_q,j_{qq})}[P_{12}, P_{22}]_{2\bullet}$ of all Arov-normalized $J_q - j_{qq}$-elementary polynomials B of formal degree m such that $B_{12} = P_{12}$ and $B_{22} = P_{22}$. In particular, characterize the situation $\mathscr{A}_m^{(J_q,j_{qq})}[P_{12}, P_{22}]_{2\bullet} \neq \emptyset$.

Theorem 3.5. *Let Q_{12} be a $q \times q$ matrix polynomial of formal degree n_1, and let Q_{22} be a $q \times q$ matrix polynomial of formal degree n_2.*

(a) *The following statements are equivalent*:

  (i) $\mathscr{A}_m^{(J_q,j_{qq})}[Q_{12}, Q_{22}]_{2\bullet} \neq \emptyset$.

  (ii) *The following conditions are satisfied*:

    (α) $n_2 < n_1$ *and* $m = n_1$.

    (β′) *Both functions* $\det \tilde{Q}_{12}^{[n_1]}$ *and* $\det Q_{22}$ *do not vanish in* $\mathbb{D} \cup \mathbb{T}$.

    (γ′) *Both matrices* $\tilde{Q}_{12}^{[n_1]}(0)$ *and* $Q_{22}(0)$ *are positive Hermitian.*

    (δ′) $Q_{12}(0) = 0$.

    (η′) *For all* $z \in \mathbb{T}$, *the identity* $(Q_{22}(z))^*Q_{22}(z) = (Q_{12}(z))^*Q_{12}(z)$ *holds.*

(b) *Let conditions* (α), (β′), (γ′), (δ′) *and* (η′) *be fulfilled. For every Hermitian* $q \times q$ *complex matrix* H, *there is a unique function* $B \in \mathscr{A}_m^{(J_q,j_{qq})}[Q_{12}, Q_{22}]_{2\bullet}$ *such that*

$$\mathrm{Im}((B_{22}(0))^{-1}B_{21}(0)) = H. \tag{3.11}$$

Proof. First assume that (i) holds. In view of

$$\mathscr{A}_m^{(J_q,j_{qq})}[Q_{12},Q_{22}]_{2\bullet} = \left(\mathscr{A}_m^{(J_q,j_{qq})}[Q_{12}]_{12}\right) \cap \left(\mathscr{A}_m^{(J_q,j_{qq})}[Q_{22}]_{22}\right)$$

we see from Theorems 3.1 and 3.3 that (α), (β′), (γ′) and (δ′) are satisfied. Let $D \in \mathscr{A}_m^{(J_q,j_{qq})}[Q_{12},Q_{22}]_{2\bullet}$. Setting $P_{12} := \check{Q}_{21}$ and $P_{22} := \check{Q}_{22}$ we see from Remark 2.3 that $C := \check{D}$ belongs to $\mathscr{A}_m^{(j_{qq},J_q)}[P_{12},P_{22}]_{2\bullet}$. According to Theorem 3.4, conditions (β), (γ), (δ) and (η) stated there are fulfilled. Then (η′) follows from (η). Thus we have proved that (ii) is necessary for (i). Conversely now assume that (ii) is satisfied. Further, let $H \in \mathbb{C}^{q\times q}$ be Hermitian. We set again $P_{21} := \check{Q}_{12}$ and $P_{22} := \check{Q}_{22}$. From (ii) we see immediately that det P_{22} has no zeros in $\mathbb{D}\cup\mathbb{T}$, that $P_{22}(0)$ is positive Hermitian and that $P_{21}(0) = 0_{q\times q}$. Moreover, condition (η′) implies (3.9) for each $z \in \mathbb{T}$. Setting $R := \tilde{Q}_{12}^{[n_1]}$ we have $\tilde{P}_{21}^{[n_1]} = \check{R}$. Therefore conditions (β′) and (γ′) show that det $\tilde{P}_{21}^{[n_1]}$ does not vanish in $\mathbb{D}\cup\mathbb{T}$ and that the matrix $\tilde{P}_{21}^{[n_1]}(0)$ is positive Hermitian. Consequently, statement (ii) in Theorem 3.4 is satisfied. Thus the application of Theorem 3.4 yields that there is a function $A \in \mathscr{A}_m^{(j_{qq},J_q)}[P_{21},P_{22}]_{2\bullet}$ which satisfies $\operatorname{Im}(A_{12}(0)(A_{22}(0))^{-1}) = -H$. Using Remark 2.3 and (1.2) we obtain that $B := \check{A}$ belongs to $\mathscr{A}_m^{(J_q,j_{qq})}[Q_{12},Q_{22}]_{2\bullet}$ and satisfies

$$\operatorname{Im}\big((B_{22}(0))^{-1}B_{21}(0)\big) = \operatorname{Im}\big((A_{22}(0))^{-*}(A_{12}(0))^*\big) = H.$$

On the other hand, we know from $\mathscr{A}_m^{(J_q,j_{qq})}[Q_{12},Q_{22}]_{\bullet 2} \subseteq \mathscr{A}_m^{(J_q,j_{qq})}[Q_{22}]_{22}$ and Corollary 2.8 that there is only one function $B \in \mathscr{A}_m^{(J_q,j_{qq})}[Q_{12},Q_{22}]_{\bullet 2}$ which satisfies (3.11). Hence part (b) is proved. In particular, we have also verified that (ii) implies (i). The proof is complete. □

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